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Quantum electrodynamic uncertainty relations for magnetic flux measurements in circuits

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Abstract. The magnetic flux sensitivity in electrical circuits is discussed in the time domain. It is shown that the quantum electrodynamic uncertainty principle implies that measurements of the magnetic flux at two different times interfere. Thus, an initial measurement of flux, at time t_1, to within $\Delta\Phi(t_1)$ will induce a random back EMF which will affect the uncertainty $\Delta\Phi(t_2)$ of the flux measurement at a later time t_2. We derive the 'interference' rule that $\Delta\Phi(t_1)\Delta\Phi(t_2) \geqslant (\hbar/2)|G(t_1, t_2)|$, where $G(t_1, t_2)$ describes the transient circuit response in flux to a weak current source impulse. As an example, this rule is applied to the case of a current-fed, singly connected Josephson weak link to give an expression for the minimum measurement uncertainty in the weak link critical current.

There has been considerable recent interest, both theoretical and experimental, in quantum electrodynamic circuits (Widom *et al* 1981, Prance *et al* 1981). In a previous letter (Widom and Clark 1980), measurements of voltage noise (in the frequency domain) were discussed in terms of power amplifiers acting as photon counters. Here, time domain quantum electrodynamic uncertainty principle restrictions on magnetic flux Φ measurements will be of interest. The physical picture is as follows. (i) If at a time t_1 a measurement of the flux $\Phi(t_1)$ in an inductor is made to within $\Delta\Phi(t_1)$, then the result is a back EMF which by the rules of quantum electrodynamics is uncertain. (ii) At a later time t_2, a circuit transient Green function $G(t_2, t_1)$ determines the effect of the back EMF on the precision $\Delta\Phi(t_2)$ of the next inductor flux measurement. (iii) The precise mathematical statement of the time domain uncertainty principle is that

$$\Delta\Phi(t_1)\Delta\Phi(t_2) \geqslant (\hbar/2)|G(t_2, t_1)|, \tag{1}$$

where the engineering prescription for measuring $G(t_2, t_1)$ as well as the mathematical definition will be given in what follows.

The general mathematical statement of the quantum uncertainty principle asserts that if a and b are two physical quantities with a commutator

$$[a, b] = \mathrm{i}\hbar c, \tag{2a}$$

then measurements of both a and b interfere with an uncertainty product

$$\Delta a\,\Delta b \geqslant (\hbar/2)|\langle c\rangle|. \tag{2b}$$

As applied to the measurement of magnetic flux in an inductor at two different times, it is evident from equations (2) that equation (1) holds if $G(t_2, t_1)$ is taken to be (Källen 1960)

$$G(t_2, t_1) = (i/\hbar)\langle[\Phi(t_2), \Phi(t_1)]\rangle. \tag{3}$$

The engineering meaning of $G(t_2, t_1)$, as defined in equation (3), can now be considered.

In figure 1(*a*) is shown (schematically) a general measurement circuit for determining the flux in an inductor L. The measuring circuit can be both nonlinear and active, i.e. it is arbitrary. In figure 1(*b*) is shown an identical measurement circuit with an added current source. If the current source is infinitesimally small, $\delta I(t)$, then the quantum averaged current source contribution to the magnetic flux $\delta\bar{\Phi}(t)$, over and above that which is due to external magnetic flux and the measurement circuit currents in figure 1(*a*), defines $G(t_2, t_1)$ as the inductor–measuring circuit transient Green function

$$\delta\bar{\Phi}(t_2) = \int_{-\infty}^{t_2} G(t_2, t_1)\delta I(t_1)\, dt_1. \tag{4}$$

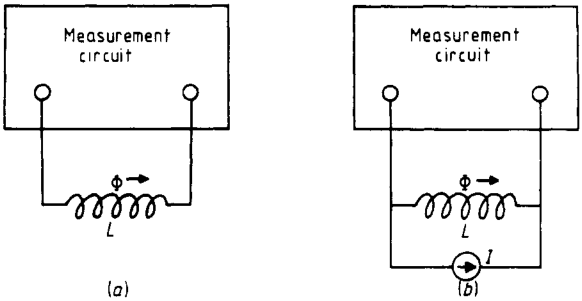

Figure 1. To obtain the inductor–measurement circuit transient function $G(t_2, t_1)$ for the uncertainty principle product $\Delta\Phi(t_1)\Delta\Phi(t_2) \geq (\hbar/2)|G(t_2, t_1)|$ in (*a*), one must consider an identical circuit system with an added current source $I(t)$ as shown in (*b*). The measurement circuit can be both nonlinear and active.

For the special case of a measurement circuit uniform in time,

$$G(t_2, t_1) = g(t_2 - t_1), \tag{5}$$

the measurement circuit–inductor total system can be regarded as having a dynamical differential inductance at frequency ζ

$$L(\zeta) = \int_0^{\infty} g(t)\, e^{i\zeta t}\, dt. \tag{6}$$

Presuming dissipation with the finite limits

$$L = \lim_{\omega \to 0} \operatorname{Re} L(\omega + i0^+), \tag{7}$$

$$\tau = \lim_{\omega \to 0} [\operatorname{Im} L(\omega + i0^+)/\omega L], \tag{8}$$

τ sets an overall timescale for the decay of the transients via the internal resistance R of the measuring device,

$$\tau = (L/R). \tag{9}$$

For a circuit in thermal equilibrium at temperature T, the symmetrised spectral distribution of inductance flux noise obeys the Nyquist theorem

$$S_\Phi(\omega) = (\hbar/2\pi)\coth(\hbar\omega/2k_BT)\,\mathrm{Im}\,L(\omega + i0^+), \tag{10}$$

so that the transient timescale determines the magnetic flux energy sensitivity (in quantum units), i.e.

$$\hbar\alpha_T = \lim_{\omega\to 0}(S_\Phi(\omega)/L) = (k_BT/\pi)\tau, \tag{11}$$

follows from equations (8) and (10). In the high-temperature region $k_BT \gg (\pi\hbar/\tau)$, i.e. $\alpha_T \gg 1$, the flux noise on the timescale τ may be regarded as classical, while in the low-temperature regime $k_BT \ll (\pi\hbar/\tau)$, i.e. $\alpha_T \ll 1$, the flux fluctuations on the timescale τ are quantum mechanical. Since equation (11) survives the generalisation from thermal temperatures T to Schwinger noise temperatures T^* (Schwinger 1960), i.e.

$$\alpha = (k_BT^*/\pi)(\tau/\hbar), \tag{12}$$

the notion that $\alpha \ll 1$ violates the uncertainty principle is often stated but not readily proven. As an example of a case where α can be much less than one, consider the circuit depicted in figure 2. For a circuit temperature T it is clear from equations (9) and (11) that $\alpha_T = (k_BT/\pi)(L/\hbar R)$. Thus, it follows that if $T \ll (\pi/k_B)\hbar R/L$ then $\alpha_T \ll 1$.

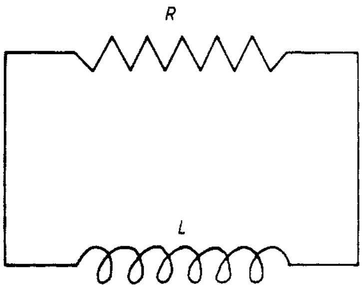

Figure 2. The transient function for a simple RL circuit has a flux energy sensitivity $\hbar\alpha_T = (k_BTL/\pi R)$ which can obey $\alpha_T \ll 1$ with an uncertainty product transient $\Delta\Phi(t_1)\Delta\Phi(t_2) \geq (\hbar R/2)\exp(-R|t_2 - t_1|/L)$.

It is a simple matter to prove uncertainty relations between quantities at given times via equation (1). Thus for the RL circuit of figure 2, equations (5) and (6) apply so that

$$g(t) = \mathrm{sgn}(t)R\exp(-|t|/\tau), \tag{13}$$

hence

$$\Delta\Phi(t_2)\Delta\Phi(t_1) \geq (\hbar R/2)\exp(-R|t_2 - t_1|/L), \qquad t_2 > t_1 \tag{14}$$

for such a circuit.

For nonlinear measuring circuits the calculation of $G(t_2, t_1)$ has all of the complications inherent to quantum electrodynamic field theoretical photon propagator renormalisations. Some insights into quantum uncertainties can be gained by computing $G(t_2, t_1)$ in the one-loop (quasi-classical) approximation. This quantum field theoretical version of the WKB method determines the uncertainty product $\Delta\Phi(t_2)\Delta\Phi(t_1)$ to lowest order in $\hbar$. In practice, the one-loop approximation consists of the evaluation of $G(t_2, t_1)$ by treating equation (4) as a classical linear response circuit transient.

An interesting case in point is the measurement of Josephson critical currents in tunnel junctions (Voss and Webb 1981), as shown in figure 3, for strongly dissipative superconducting weak links. The current source is set below the critical current I_c, i.e.

$$\Delta I = (I_c - I) > 0, \tag{15}$$

and ΔI (when gradually decreased) is considered to vanish when the link undergoes a transition to a voltage $V \simeq RI_c$ carrying state. As the critical current is approached the flux must be determined at least to the extent that

$$\Delta\Phi^2 \leqslant (\hbar^2/8e^2)(\Delta I/I_c). \tag{16}$$

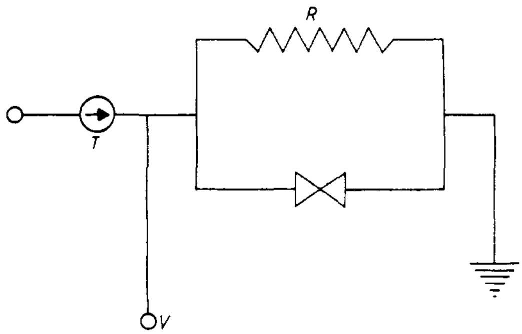

Figure 3. Conventional *RSJ* model circuit for measuring Josephson critical currents in a strongly dissipative weak link.

On a timescale short compared with long damping times but long on the scale of strongly dissipative rise times, equation (14) holds true in the form

$$\Delta\Phi^2 \geqslant (\hbar R/2). \tag{17}$$

Equations (16) and (17) yield a quantum limitation on the method of measuring I_c outlined above. It is

$$(\Delta I/I_c) \geqslant 4(e^2/\hbar)R \simeq (10^{-3}R/\text{ohms}). \tag{18}$$

Equation (18), which holds only for strongly overdamped weak links, states that for a given value of R there is an uncertainty principle restriction on the accuracy to which the junction critical current I_c can be measured. In the example quoted above (Voss and Webb 1981) the experimental data are analysed in terms of a WKB model of a Josephson junction (Caldeira and Leggett 1981). According to our calculations the measured critical current will be set by the experimental value chosen for R. It is common practice (see Voss and Webb 1981) to equate R with the asymptotic slope resistance of the junction at bias currents large compared with I_c. At such bias currents the internal Josephson frequencies in the junction are comparable to the superconducting gap frequency. Such high frequencies mean (i) pair breaking must occur at the junction and (ii) due to the finite capacitance of a Josephson tunnel junction the shunt conductance of the junction at frequencies ~ the gap frequency must be very high. Both these effects lead us to believe that the asymptotic slope resistance should not be equated with the R of equation (18) for the purpose of predicting the minimum measurement uncertainty in I_c. In this sense, the quantum limitation on the accuracy of measurement of I_c depends on the method of measurement used.

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